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The Atiyah–Singer index theorem and the gauge field copy problem*

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Abstract. K-theory allows us to define an analytical condition for the existence of 'false' gauge field copies through the use of the Atiyah–Singer index theorem. After establishing this result we discuss a possible extension of the same result without the help of the index theorem and suggest possible related lines of work.

1. Introduction

The gauge field copy question is one of those unexpected phenomena that creep up in mathematical physics when we go from linear objects to their nonlinear extensions. Linear gauge fields, say, the electromagnetic field, admit a single potential over a nice neighbourhood modulo gauge tranformations. However, when we go from the linear to the nonlinear domain, that nice relation between fields and potentials breaks down. (We emphasize the adjective: the relation between potentials and fields in linear gauge fields is a 'nice' one because it reflects the very deep $\partial^2 = 0$ relation in homological algebra and in algebraic topology; for a simple application of that relation to mathematical physics see [10].)

In the nonlinear, non-Abelian case, some gauge fields admit two or more potentials which cannot be made equivalent (even locally) modulo gauge transformations. Such an ambiguity is known as the gauge field copy problem and was discovered in 1975 by Wu and Yang [17]. Gauge copies fall into two cases.

- *True copies*. Here the gauge field can be derived from at least two different potentials that are not even locally related by a gauge transformation.
- *False copies*. In this case the field can be derived from potentials that are always locally related by a gauge transformation.

For a review of the geometric phenomena behind the copy problem see [6].

Let $P(M, G)$ be a principal fibre bundle, where M is a finite-dimensional smooth real manifold and G is a finite-dimensional semi-simple Lie group. We denote by (P, α) the principal fibre bundle $P(M, G)$ endowed with the connection form α, and by L the field that corresponds to the potential A associated with α. We mean by this that A is a family

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$\{A_\lambda\}_{\lambda\in\Lambda}$ of $l(G)$-valued 1-forms, where $l(G)$ is the Lie algebra associated with G and, for each $\lambda \in \Lambda$, A_λ is defined on an open subset U_λ of M with $D_{A_\lambda} = dA_\lambda + \frac{1}{2}[A_\lambda, A_\lambda] = L$ on U_λ and with $M = \bigcup_{\lambda\in\Lambda} U_\lambda$. We consider also that A_λ and $A_{\lambda'}$ are gauge equivalent for $\lambda \neq \lambda'$. We recall that an automorphism of a principal fibre bundle $P(M, G)$ with projection π is a diffeomorphism $f : P \to P$ such that $f(pg) = f(p)g$, for all $g \in G$, $p \in P$. A gauge transformation is an automorphism $f : P \to P$ such that $\overline{f} = 1_M$, where $\overline{f} : M \to M$ is the diffeomorphism induced by f given by $\overline{f}(\pi(p)) = \pi(f(p))$. Given a principal fibre bundle $P(M, G)$ and a Lie subgroup G' of G, we say that $P(M, G)$ is reducible to the bundle $P'(M, G')$ if and only if there is a monomorphism $f' : G' \to G$ and an imbedding $f'' : P' \to P$ such that $f''(u'a') = f''(u')f'(a')$, for all $u' \in P'$ and $a' \in G'$.

Definition 1.1. (1) The field L or the potential A are said to be *reducible* if the corresponding bundle (P, α) is reducible.

(2) If $U \subset M$ is a non-void open set then L or A are said to be *locally reducible over* U whenever $(P, \alpha)|_U$ is reducible.

(3) L or A are said to be *fully irreducible* if they are not locally reducible. □

Our main results are essentially based on a theorem that gives a topological condition for the existence of false copies and on the Atiyah–Singer index theorem. However, before we state the topological condition for the existence of false gauge field copies, we find it interesting to recall the Ambrose–Singer theorem, since it will be used to derive such a result.

Proposition 1.1. (Ambrose–Singer). Let $P(M, G)$ be a principal fibre bundle, where M is connected and paracompact. Let Γ be a connection in P, L the curvature form, $\Phi(u)$ the holonomy group with reference point $u \in P$ and $P(u)$ the holonomy bundle through u of Γ. Then the Lie algebra of $\Phi(u)$ is equal to the subspace of $l(G)$ spanned by all elements of the form $L_v(X, Y)$, where $v \in P(u)$ and X and Y are arbitrary horizontal vectors at v.

The proof of this theorem (also known as the holonomy theorem), can be found in [12]. It is considered that we may assume $P(u) = P$, which means that $\Phi(u) = G$.

Now we establish the topological condition for the existence of false gauge field copies, based on a result due to Doria [8].

Proposition 1.2. Let $P(M, G)$ be as above together with the fact that G be semi-simple and M is connected and paracompact. L is falsely copied, that is, L has different potentials that are locally related by a gauge transformation if and only if L is reducible.

Proof. If L is reducible, then $P(M, G)$ is reducible (definition 1.1). This means that $P(M, G)$ can be reduced to a non-trivial $P'(M, G')$, where G' is the Ambrose–Singer holonomy group (it corresponds to the group $\Phi(u)$ in proposition 1.1). If we assume

$$A_\mu = B_\mu + \partial_\mu h'$$

where $\partial_\mu =_{\mathrm{def}} \partial/\partial x^\mu$, x^μ is a coordinate of a coordinate system at $U \subset M$ (such that the bundle is trivial over U), B_μ takes values in $l(G)$ and h' takes values on $l(G')$, then

$$F_{\mu\nu}(A) = \partial_\mu(B_\nu + \partial_\nu h') - \partial_\nu(B_\mu + \partial_\mu h') + (B_\mu + \partial_\mu h')(B_\nu + \partial_\nu h') \\ -(B_\nu + \partial_\nu h')(B_\mu + \partial_\mu h')$$

where $F_{\mu\nu}(A)$ denotes the components of the curvature form F associated with the connection form A.

Hence,

$$F_{\mu\nu}(A) = F_{\mu\nu}(B) + 0_{\text{field}}.$$

The sufficient condition is proved as follows: if the holonomy group is semi-simple as indicated, and if $A_\mu = B_\mu + \partial_\mu h'$ as indicated, then there is a reducibility $G \oplus G' \to G'$. □

We now suppose that X is a compact smooth manifold and that G is a compact Lie group acting smoothly on M. The Atiyah–Singer index theorem can be stated as follows [16].

Proposition 1.3. Let χ and ϑ be complex vector bundles defined over X. If $D : C^\infty(X;\chi) \to C^\infty(X;\vartheta)$ is a G-invariant elliptic partial differential operator on X, which sends cross sections of χ to cross sections of ϑ, then index ${}_G D = t_{\text{ind}_G^X}(\sigma(D))$, where t_{ind} is the topological index defined on $K_G(TX)$ and $\sigma(D)$ is the symbol of D.

(For the proof and notational features see [16].)

Another version of the index theorem [4] asserts the following.

Proposition 1.4. The analytical index a_{ind_G} and the topological index $t_{\text{ind}_G^X}$ coincide as homomorphisms $K_G(TX) \to R(G)$.

(Proof in [4].)

2. A necessary condition for false copies

Gauge fields and gauge potentials can be seen and defined as cross sections of vector bundles associated with the principal fibre bundle $P(M, G)$. More specifically, potential space (or connection space) coincides with the space of all C^k cross sections of the vector bundle E of $l(G)$-valued 1-forms on M, where $l(G)$ is the group's Lie algebra, while field space (or curvature space) coincides with the space of all C^k cross sections of the vector bundle $\boldsymbol{E}$ of $l(G)$-valued 2-forms on M.

Let F and $\boldsymbol{F}$ be manifolds on which G acts on the left and such that $E = P \times_G F$ and $\boldsymbol{E} = P \times_G \boldsymbol{F}$, where P is the total space of $P(M, G)$. In other words, E is the quotient space of $P \times F$ by the group action. Similarly, $\boldsymbol{E}$ is the quotient space of $P \times \boldsymbol{F}$ by the action of the group G.

To prove the following proposition we use the Atiyah–Singer index theorem. So, we are still assuming that M and G are compact.

Therefore:

Proposition 2.1. If a gauge field (a cross section of $\boldsymbol{E}$) is associated with copied potentials that are locally gauge equivalent, then there is: a non-trivial sub-group of G, denoted by G'; a G'-manifold P'; two G'-vector spaces F' and $\boldsymbol{F}'$; and two elliptic partial differential operators,

$$\mathcal{D}_G : C^\infty(P; P \times F) \to C^\infty(P; P \times \boldsymbol{F}) \tag{1}$$

and

$$\mathcal{D}_{G'} : C^\infty(P'; P' \times F') \to C^\infty(P'; P' \times \boldsymbol{F}') \tag{2}$$

respectively G-invariant and G'-invariant, such that the index $\mathcal{D}_{G'}$ can be defined as a non-trivial function of the index $\mathcal{D}_G$.

Proof. If a gauge field is associated with copied potentials that are locally gauge equivalent, then such a field is reducible (proposition 1.2). Therefore, the bundle $P(M,G)$ is reducible (definition 1.1). So, there is a non-trivial subgroup G' of G and a monomorphism $\varphi : G' \to G$ such that one can define a reduced principal fibre bundle $P'(M,G')$ and a reduction $f : P'(M,G') \to P(M,G)$. Similarly we define G'-vector spaces F' and $\boldsymbol{F}'$ and maps $\mathbf{f} : \boldsymbol{F}' \to \boldsymbol{F}$ and $\mathrm{f} : F' \to F$ such that

$$\mathrm{f}(g'\xi') = \varphi(g')\mathrm{f}(\xi') \tag{3}$$

and

$$\mathbf{f}(g'\zeta') = \varphi(g')\mathbf{f}(\zeta') \tag{4}$$

for all $g' \in G'$, $\xi' \in F'$ and $\zeta' \in \boldsymbol{F}'$.

Now consider $P \times F$ as the total space of the trivial vector bundle $P \times F \to P$, with a canonical projection. That vector bundle is noted χ. Similarly the trivial vector bundles $P \times \boldsymbol{F} \to P$, $P' \times F' \to P'$ and $P' \times \boldsymbol{F}' \to P'$ are noted ϑ, χ' and ϑ', respectively.

Therefore, the diagram below commutes:

$$\begin{array}{ccc} K_G(TP) & \xrightarrow{\varphi^*} & K_{G'}(TP') \\ t_{\mathrm{ind}_G^P} \downarrow & & \downarrow t_{\mathrm{ind}_{G'}^{P'}} \\ R(G)) & \xrightarrow{\varphi^*} & R(G') \end{array}$$

(Here φ^* is induced by φ.)

$\sigma(\mathcal{D}_G) = \vartheta - \chi$ and $\sigma(\mathcal{D}_{G'}) = \vartheta' - \chi'$. Therefore, the homomorphisms f, $\mathbf{f}$ and f and the diagram given above induce the relation

$$\sigma(\mathcal{D}_{G'}) = \varphi^*(\sigma(\mathcal{D}_G)). \tag{5}$$

Thus, according to the diagram, $t_{\mathrm{ind}_{G'}^{P'}}(\sigma(\mathcal{D}_{G'})) = \varphi^*(t_{\mathrm{ind}_G^P}(\sigma(\mathcal{D}_G)))$. If we use the Atiyah–Singer index theorem 1.3, it can be noted that

$$\text{index } \mathcal{D}_{G'} = \varphi^*(\text{index } \mathcal{D}_G). \tag{6}$$

□

Remark 2.1. The condition that $\mathcal{D}_G$ and $\mathcal{D}_{G'}$ are elliptic partial differential operators is necessary in our proof of proposition 2.1 in order to apply the Atiyah–Singer index theorem given by proposition 1.3.

The topological condition given in [8], in order to check whether there are false gauge field copies does not impose that the manifold M should be compact, or that the Lie group must be compact (a very common situation in gauge theories). However, when we apply the Atiyah–Singer index theorem to obtain the analytical condition for false copies, we must consider that both M and G are compact.

3. Conclusions

There are several points of contact between classical physics and K-theory: a method to prove the index theorem based on the asymptotics of the heat equation [3, 11]; and the physical interpretation of non-vanishing characteristic classes in terms of magnetic monopoles, solitons and instantons [13]. We present here a new K-theoretical result with a consequence to physics: an analytical condition to the existence of 'false' gauge field copies obtained from a topological condition.

Our result has some limitations: it refers only to false copies, it is imposed that M and G are compact; and it is imposed that G is semi-simple. We believe that it is possible to extend our results while eliminating those restrictions. (One possibility should be to modify the geometry of an irreducible principal fibre bundle in such a way as to handle true copies as false copies (see [6] on this).) That is, a possible way to define an analytical condition for generic copies through the use of the Atiyah–Singer index theorem. On the structure of M, we just recall that K-theory has a technique for dealing with locally compact spaces. Moreover, a possible way of generalizing our results to any Lie group (compact or non-compact) amounts (we hope) to the use of some kind of compactification [7].

Another side result of the present work has to do with the categorical approach that underlies K-theory. When we formalize gauge theory and the gauge copy problem with tools from category theory we note that our categorical formalization ends up as being too complicated when compared with the traditional approach based on set theory [5].

Moreover, we think it is possible to obtain the same result of proposition 2.1 without K-theoretical concepts. First, these results make no reference to K-theoretical elements. The index theorem only appears in the proof of the propositions. Second, there is an analytical proof (non-topological) of the index theorem by the use of Zeta functions $\zeta(s) = \sum \lambda^{-s}$ (λ denotes eigenvalues of an operator) [2]. Therefore, we conjecture that it may be possible to prove the analytical condition that we have established for the existence of false gauge field copies in a similar way.

Still another possibility has to do with proving our results with the help of the asymptotics of the heat equation [11], since that technique was also used to prove the index theorem. However, these are questions to be answered in future papers.

All these questions can be summed up in the following. The gauge field copy phenomenon is a remarkable feature of nonlinear gauge fields without a clear-cut physical interpretation. Copied fields belong to a nowhere dense, bifurcation-like domain in the space of all gauge fields in the usual topology [9]. Copied fields are actually involved in a (possible) symmetry-breaking mechanism that remains strictly within the bounds of classical field theory [1], yet their contribution to path-integral computations remain unknown—for instance, if Feynmann integrals are understood as Kurtzweil–Henstock (KH) integrals [14, 15], the rather unusual topology of the space of all KH-integrable functions may lead to a new interpretation of the Feynman integrals over field and potential spaces in gauge theory, when given the adequate KH-structure.

Therefore, when we try to establish a connection between Atiyah–Singer theory and gauge copies, we are trying to expand the perspectives from where we can observe and try to understand the field copy phenomenon.

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References

[1] Furtado do Amaral A F, Doria F A and Gleiser M 1983 *J. Math. Phys.* **24** 1888

[2] Atiyah M F and Bott R 1967 *Ann. Math.* **86** 374

[3] Atiyah M F, Bott R and Patodi V K 1973 *Invent. Math.* **19** 279
See the errata in Atiyah M F, Bott R and Patodi V K 1975 *Invent. Math.* **28** 277

[4] Atiyah M F and Singer I M 1968 *Ann. Math.* **87** 484–530

[5] da Costa N C A and Doria F A 1992 Suppes predicates for classical physics *The Space of Mathematics* ed J Echeverría *et al* (Berlin–New York: Walter de Gruyter)

[6] da Costa N C A, Doria F A, Furtado do Amaral A F and de Barros J A 1994 *Found. Phys.* **24** 783

[7] Dikranjan D N, Prodanov I R and Stoyanov L N 1989 *Topological Groups—Characters, Dualities, and Minimal Group Topologies* (New York: Dekker)

[8] Doria F A 1981 *Comm. Math. Phys.* **79** 435

[9] Doria F A 1984 *Functional Analysis, Holomorphy and Approximation Theory II* ed G I Zapata (Amsterdam: North-Holland) p 69

[10] Doria F A and Abrahão S M 1978 *J. Math. Phys.* **19** 1650

[11] Gilkey P B 1974 *The Index Theorem and the Heat Equation (Mathematical Lecture Series 4)* (Boston: Publish or Perish)

[12] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* (New York: Wiley)

[13] Mayer M E 1981 *Gauge Theories, Vector Bundles and the Index Theorem* (Berlin: Springer)

[14] Muldowney P 1987 *A General Theory of Integration in Function Spaces* (London: Longman)

[15] Peng-Yee L 1989 *Lanzhou Lectures on Henstock Integration* (Singapore: World Scientific)

[16] Shanahan P 1978 *The Atiyah–Singer Index Theorem, an Introduction (Lecture Notes in Mathematics 638)* (Berlin: Springer)

[17] Wu T T and Yang C N 1975 Some remarks about unquantized non-Abelian gauge fields *Phys. Rev.* D **12** 3843